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Boyd indices for quasi-normed function spaces with some bounds

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available at the end of the article**Abstract**

We calculate the Boyd indices for quasi-normed rearrangement invariant function spaces with some bounds. An application to Lorentz type spaces is also given.

MSC: 46E30; 46E35**Keywords:** rearrangement invariant function spaces; Boyd indices; quasi-normed function spaces

1 Introduction

Let L_{loc} be the space of all locally integrable functions f on \mathbf{R}^n and M^+ be the cone of all locally integrable functions $g \geq 0$ on $(0, 1)$ with the Lebesgue measure.

Let f^* be the decreasing rearrangement of f given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t > 0,$$

and μ_f be the distribution function of f defined by

$$\mu_f(\lambda) = |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|_n,$$

$|\cdot|_n$ denoting Lebesgue n -measure.

Also,

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) ds.$$

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

We consider rearrangement invariant quasi-normed spaces $E \hookrightarrow L^1(\Omega)$ such that $\|f\|_E = \rho_E(f^*) < \infty$, where ρ_E is a quasi-norm rearrangement invariant defined on M^+ .

For simplicity, we assume that Ω is a bounded Lebesgue measurable subset of \mathbf{R}^n with Lebesgue measure equal to 1 and origin lies in Ω .

There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0, 1]$ that depends only on the space E (see [1]). We say

that the quasi-norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p ,

$$\rho_p^p\left(\sum g_j\right) \lesssim \sum \rho_p^p(g_j), \quad g_j \in M^+.$$

Usually we apply this inequality for functions $g \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E . Let $g_u(t) = g(t/u)$ if $t < \min(1, u)$ and $g_u(t) = 0$ if $\min(1, u) < t < 1$, where $g \in M^+$, and let

$$h_E(u) = \sup \left\{ \frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+ \right\}, \quad u > 0$$

be the dilation function generated by ρ_E . Suppose that it is finite. Then

$$\alpha_E := \sup_{0 < t < 1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1 < t < \infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is sub-multiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$ and hence $0 \leq \alpha_E \leq \beta_E$. We suppose that $0 < \alpha_E = \beta_E \leq 1$.

If $\beta_E < 1$, we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(f^{**})$. In particular, $\|f\|_E \approx \rho_E(f^{**})$ if $\beta_E < 1$. For example, consider the gamma spaces $E = \Gamma^q(w)$, $0 < q \leq \infty$, w -positive weight, that is, a positive function from M^+ , with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_E(f^*)$, $\rho_E(g) := \rho_{w,q}(\int_0^1 g(tu) du)$, where

$$\rho_{w,q}(g) := \left(\int_0^1 [g(t)w(t)]^q dt/t \right)^{1/q}, \quad g \in M^+ \quad (1.1)$$

and

$$\left(\int_0^1 w^q(t) dt/t \right)^{1/q} < \infty.$$

Then $L^\infty(\Omega) \hookrightarrow \Gamma^q(w) \hookrightarrow L^1(\Omega)$. If $w(t) = t^{1/p}$, $1 < p < \infty$, we write as usual $L^{p,q}$ instead of $\Gamma^q(t^{1/p})$. Consider also the classical Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$; $f \in \Lambda^q(w)$ if $\|f\|_{\Lambda_w^q} := \rho_{w,q}(f^*) < \infty$, $w(2t) \approx w(t)$. We suppose that $L^\infty(\Omega) \hookrightarrow \Lambda^q(w) \hookrightarrow L^1(\Omega)$.

The Boyd indices are useful in various problems concerning continuity of operators acting in rearrangement invariant spaces [2] or in optimal couples of rearrangement invariant spaces [3–5], and in the problems of optimal embeddings [6–8]. The main goal of this paper is to provide formulas for the Boyd indices with some bounds of rearrangement invariant quasi-normed spaces and to apply these results to the case of Lorentz type spaces.

2 Boyd indices for quasi-normed function spaces

Let ρ_E be a monotone quasi-norm on M^+ and let E be the corresponding rearrangement invariant quasi-normed space consisting of all $f \in L^1(\Omega)$ such that $\|f\|_E = \rho_E(f^*) < \infty$.

Theorem 2.1 *Let*

$$g_u(t) = \begin{cases} g(t/u) & \text{if } 0 < t < \min(1, u), \\ 0 & \text{if } \min(1, u) \leq t < 1, \end{cases}$$

where $g \in M^+$, and let

$$h_E(u) = \sup \left\{ \frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+ \right\}, \quad u > 0,$$

be the dilation function generated by ρ_E . Suppose that it is finite. Then the Boyd indices are well defined

$$\alpha_E := \sup_{0 < t < 1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1 < t < \infty} \frac{\log h_E(t)}{\log t}$$

and they satisfy

$$\alpha_E = \lim_{t \rightarrow 0} \frac{\log h_E(t)}{\log t}, \tag{2.1}$$

$$\beta_E = \lim_{t \rightarrow \infty} \frac{\log h_E(t)}{\log t}. \tag{2.2}$$

In particular, $0 \leq \alpha_E \leq \beta_E \leq \frac{\log h_E(2)}{\log 2}$.

Proof We have

$$g_{uv} = (g_u)_v \quad \text{if } u < v. \tag{2.3}$$

Indeed, since $\min(1, uv) \leq \min(1, v)$ for $u < v$, we find $(g_u)_v(t) = g_u(t/(uv))$ if $0 < t < \min(1, uv)$ and $(g_u)_v(t) = 0$ if $\min(1, uv) \leq t < 1$. Thus (2.3) is proved. This implies that the function h_E is sub-multiplicative.

Further, the function $\omega(x) = \log h_E(e^x)$ is sub-additive increasing on $(-\infty, \infty)$ and $\omega(0) = 0$. Hence, by [2], Lemma 5.8, (2.2) is satisfied and evidently $\beta_E \leq \frac{\log h_E(2)}{\log 2}$.

Since $h_E(1) = 1$ and h_E is sub-multiplicative, therefore

$$h_E(u_1 u_2) \leq h_E(u_1) h_E(u_2).$$

Replacing u_2 by $\frac{1}{u_1}$, we get

$$h_E(1) \leq h_E(u_1) h_E\left(\frac{1}{u_1}\right),$$

which implies that

$$1 \leq h_E(u_1) h_E\left(\frac{1}{u_1}\right); \quad \text{because } h_E(1) = 1,$$

it follows that $1 \leq h_E(u) h_E(1/u)$.

We have

$$\alpha_E \leq \beta_E.$$

Indeed

$$\log(h_E(u)) \geq \log\left(\frac{1}{h_E(\frac{1}{u})}\right),$$

if $u > 1$, then

$$\frac{\log(h_E(u))}{\log u} \geq \frac{\log(\frac{1}{h_E(\frac{1}{u})})}{\log u} = \frac{\log(h_E(\frac{1}{u}))}{\log \frac{1}{u}},$$

which implies that

$$\lim_{u \rightarrow \infty} \frac{\log(h_E(u))}{\log u} \geq \lim_{u \rightarrow \infty} \frac{\log(h_E(\frac{1}{u}))}{\log \frac{1}{u}}.$$

Since β_E is finite, therefore α_E is also finite. Since $h_E(1) = 1$ and we know that h_E is increasing function, so

$$h_E(u) \leq 1 \quad \text{for } 0 < u < 1,$$

which implies that

$$\log(h_E(u)) \leq 0,$$

which implies that

$$\frac{\log(h_E(u))}{\log u} \geq 0,$$

which implies that

$$\alpha_E = \sup_{0 < u < 1} \frac{\log(h_E(u))}{\log u} \geq 0,$$

and hence

$$0 \leq \alpha_E \leq \beta_E.$$

□

Let ρ_H be a monotone quasi-norm on M^+ and let H be the corresponding quasi-normed space, consisting of all locally integrable functions on $(0, 1)$ with a finite quasi-norm $\|g\|_H = \rho_H(|g|)$.

Theorem 2.2 *Let*

$$(\Psi_u g)(t) = \begin{cases} g(ut), & \text{if } 0 < t < \min(1, \frac{1}{u}), \\ g(1), & \text{if } \min(1, \frac{1}{u}) \leq t < 1, \end{cases}$$

where $g \in M^+$, and let

$$h_H(u) = \sup \left\{ \frac{\rho_H(\Psi_u g)}{\rho_H(g)} : g \in G_a \right\}, \quad u > 0,$$

be the dilation function generated by ρ_H . Suppose that it is finite, where

$$G_a := \{g \in M^+ : t^{-a/n}g(t) \text{ is decreasing}\}, \quad a > 0.$$

Then the Boyd indices are well defined

$$\alpha_H := \sup_{0 < t < 1} \frac{\log h_H(t)}{\log t} \quad \text{and} \quad \beta_H := \inf_{1 < t < \infty} \frac{\log h_H(t)}{\log t}$$

and they satisfy

$$\alpha_H = \lim_{t \rightarrow 0} \frac{\log h_H(t)}{\log t}, \quad (2.4)$$

$$\beta_H = \lim_{t \rightarrow \infty} \frac{\log h_H(t)}{\log t}. \quad (2.5)$$

In particular, $\frac{\log h_H(1/2)}{\log 1/2} \leq \alpha_H \leq \beta_H \leq a/n$.

Proof We have

$$\Psi_{uv}g = \Psi_u(\Psi_vg) \quad \text{if } u < v. \quad (2.6)$$

Indeed, since $\min(1, 1/(uv)) \leq \min(1, 1/u)$ for $u < v$, we find $\Psi_u(\Psi_vg)(t) = g(t/(uv))$ if $0 < t < \min(1, 1/(uv))$ and $\Psi_u(\Psi_vg)(t) = g(1)$ if $\min(1, 1/(uv)) \leq t < 1$. Thus (2.6) is proved. This implies that the function h_H is sub-multiplicative. Since the function $u^{-a/n}h_H(u)$ is decreasing, it follows that the function $u^{a/n}h_H(1/u)$ is increasing and sub-multiplicative. Hence we can apply the results from Theorem 2.1. This establishes Theorem 2.2. \square

Example 2.3 If $E = \Lambda^q(t^a w)$, $0 \leq a \leq 1$, $0 < q \leq \infty$, where w is slowly varying, then $\alpha_E = \beta_E = a$.

Proof We give a proof for $0 < q < \infty$, the case $q = \infty$ is analogous. We have, for $g \in M^+$,

$$\rho_E(g_u^*) = \left(\int_0^1 [g_u^*(t)t^a w(t)]^q dt/t \right)^{1/q} = \left(\int_0^{\min(1, u)} [g^*(t/u)t^a w(t)]^q dt/t \right)^{1/q}$$

and by a change of variables,

$$\rho_E(g_u^*) \leq \left(\int_0^1 [g^*(t)(tu)^a w(tu)]^q dt/t \right)^{1/q}. \quad (2.7)$$

From the definition of a slowly varying function it follows that for every $\varepsilon > 0$, $t^{-\varepsilon}w(t) \approx d(t)$, where d is a decreasing function. Then, for $u > 1$, we have $d(tu) \leq d(t)$, thus

$$(tu)^{-\varepsilon}w(tu) \lesssim d(tu) \lesssim t^{-\varepsilon}w(t),$$

which implies that

$$w(tu) \lesssim u^\varepsilon w(t), \quad u > 1. \quad (2.8)$$

Inserting this estimate in (2.7), we arrive at

$$\rho_E(g_u^*) \lesssim u^{\alpha+\varepsilon} \rho_E(g^*), \quad u > 1,$$

which yields $h_E(u) \lesssim u^{\alpha+\varepsilon}$, $u > 1$. Then it follows that $\beta_E \leq \alpha + \varepsilon$. Analogously, $\alpha_E \geq \alpha - \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\alpha_E \leq \beta_E$, we obtain $\alpha_E = \beta_E = \alpha$. \square

Example 2.4 If $H = L_*^q(w(t)t^{-\alpha})$, $0 \leq \alpha < a/n$, $0 < q \leq \infty$, where w is slowly varying, then $\alpha_H = \beta_H = \alpha$.

Proof We give a proof for $0 < q < \infty$, the case $q = \infty$ is analogous. We have, for $g \in G_a$,

$$\begin{aligned} \rho_H(\Psi_u g) &= \left(\int_0^1 [\Psi_u g(t) t^{-\alpha} w(t)]^q dt/t \right)^{1/q} \\ &= \left(\int_0^{\min(1, 1/u)} [g(tu) t^{-\alpha} w(t)]^q dt/t \right)^{1/q} + I(u), \end{aligned}$$

where $I(u) = (\int_{\min(1, 1/u)}^1 [t^{-\alpha} w(t)]^q dt/t)^{1/q} g(1)$. Note that $I(u) = 0$ for $0 < u < 1$. Since for every $\varepsilon > 0$ we have $w(t) \lesssim t^\varepsilon$, it follows that $I(u) \lesssim u^{\alpha+\varepsilon} g(1)$, $u > 1$. Also, $g(1) \rho_H(t^{a/n}) \leq \rho_H(g)$ and $\rho_H(t^{a/n}) < \infty$ due to $\alpha < a/n$.

On the other hand, by a change of variables,

$$\rho_H(\Psi_u g) \lesssim \left(\int_0^1 [g(t)(t/u)^{-\alpha} w(t/u)]^q dt/t \right)^{1/q} + u^{\alpha+\varepsilon} \rho_H(g).$$

As in the proof of the previous example, we have

$$w(t/u) \lesssim u^\varepsilon w(t), \quad u > 1,$$

therefore

$$\rho_H(\Psi_u g) \lesssim u^{\alpha+\varepsilon} \rho_H(g), \quad u > 1, g \in G_a.$$

Hence $h_H(u) \lesssim u^{\alpha+\varepsilon}$, $u > 1$. Then it follows that $\beta_H \leq \alpha + \varepsilon$. Analogously, $\alpha_H \geq \alpha - \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\alpha_H \leq \beta_H$, we obtain $\alpha_H = \beta_H = \alpha$. \square

3 Basic inequalities

Here we prove a few inequalities, which are of independent interest.

Theorem 3.1 If $\alpha < \alpha_H$, then

$$\rho_H \left(t^\alpha \int_0^t s^{-\alpha} g(s) \frac{ds}{s} \right) \lesssim \rho_H(g), \quad g \in G_a$$

and if $\beta_H < \beta$, then

$$\rho_H \left(t^\beta \int_t^1 s^{-\beta} g(s) \frac{ds}{s} \right) \lesssim \rho_H(g), \quad g \in G_a.$$

Proof We are going to use Minkowski's inequality for the equivalent p -norm of ρ_H . To this end, first we replace the integrals by sums using monotonicity properties of $g \in G_a$.

Thus

$$\begin{aligned} t^\alpha \int_0^t s^{-\alpha} g(s) \frac{ds}{s} &= \int_0^1 v^{-\alpha} g(tv) \frac{dv}{v} \\ &= \sum_{l=-\infty}^0 \int_{2^{l-1}}^{2^l} v^{-\alpha} g(tv) \frac{dv}{v} \\ &\lesssim \sum_{l=-\infty}^0 2^{-l\alpha} g(t2^l). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \rho_H^p \left(t^\alpha \int_0^t s^{-\alpha} g(s) \frac{ds}{s} \right) &\lesssim \sum_{l=-\infty}^0 2^{-lp\alpha} \rho_H^p(g(t2^l)) \\ &\lesssim \rho_H^p(g) \sum_{l=-\infty}^0 2^{-p\alpha l} h_H^p(2^l) \\ &\lesssim \rho_H^p(g) \sum_{l=-\infty}^0 2^{-p\alpha l} 2^{lp(\alpha_H - \varepsilon)} \\ &\lesssim \rho_H^p(g) \sum_{l=-\infty}^0 2^{lp(\alpha_H - \varepsilon - \alpha)}. \end{aligned}$$

The above series is convergent if we choose $\varepsilon > 0$ such that $\varepsilon < \alpha_H - \alpha$, so we have

$$\rho_H \left(t^\alpha \int_0^t s^{-\alpha} g(s) \frac{ds}{s} \right) \lesssim \rho_H(g).$$

On the other hand, for $g \in G_a$,

$$\begin{aligned} t^\beta \int_t^1 s^{-\beta} g(s) \frac{ds}{s} &= \int_1^\infty \chi_{(0,1)}(tv) v^{-\beta} g(tv) \frac{dv}{v} \\ &= \sum_{l=0}^\infty \int_{2^l}^{2^{l+1}} \chi_{(0,1)}(tv) v^{-\beta} g(tv) \frac{dv}{v} \\ &\lesssim \sum_{l=0}^\infty 2^{-l\beta} g(t2^l) \chi_{(0,1)}(t2^l). \end{aligned}$$

Again applying Minkowski's inequality, we get

$$\begin{aligned} \rho_H^p \left(t^\beta \int_t^1 s^{-\beta} g(s) \frac{ds}{s} \right) &\lesssim \sum_{l=0}^\infty 2^{-l\beta p} \rho_H^p(g(t2^l) \chi_{(0,1)}(t2^l)) \\ &\lesssim \rho_H^p(g) \sum_{l=0}^\infty 2^{-l\beta p} h_H^p(2^l) \end{aligned}$$

$$\begin{aligned} &\lesssim \rho_H^p(g) \sum_{l=0}^{\infty} 2^{-l\beta p} 2^{lp(\beta_H+\varepsilon)} \\ &\lesssim \rho_H^p(g) \sum_{l=0}^{\infty} 2^{lp(\beta_H+\varepsilon-\beta)}. \end{aligned}$$

The above series is finite if we choose a suitable $\varepsilon > 0$ such that $\varepsilon < \beta - \beta_H$. The proof is finished. \square

Theorem 3.2 *If $\beta_E < a$, then*

$$\rho_E \left(t^{-a} \int_0^t s^a g(s) \frac{ds}{s} \right) \lesssim \rho_E(g), \quad g \in D_0,$$

where $D_0 := \{g \in M^+ : g(t) \text{ is decreasing and } g(t) = 0 \text{ for } t \geq 1\}$.

Proof We are going to use Minkowski's inequality for the equivalent p -norm of ρ_E . To this end, first we replace the integral by sums using monotonicity properties of $g \in D_0$.

Thus

$$\begin{aligned} t^{-a} \int_0^t s^a g(s) \frac{ds}{s} &= \int_0^1 v^a g(tv) \frac{dv}{v} \\ &= \sum_{l=-\infty}^0 \int_{2^l}^{2^{l+1}} v^a g(tv) \frac{dv}{v} \\ &\lesssim \sum_{l=-\infty}^0 2^{al} g(2^l). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \rho_E^p \left(t^{-a} \int_0^t s^a g(s) \frac{ds}{s} \right) &\lesssim \sum_{l=-\infty}^0 2^{pal} \rho_E^p(g(2^l)) \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^0 2^{pal} h_E^p(2^l) \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^0 2^{pal} 2^{-1p(\beta_E+\varepsilon)} \\ &\lesssim \rho_E^p(g) \sum_{l=-\infty}^0 2^{lp(a-\beta_E-\varepsilon)}. \end{aligned}$$

The above series is finite if we choose $\varepsilon > 0$ such that $\varepsilon < a - \beta_E$, and this concludes the proof. \square

Theorem 3.3 *If $\alpha_E > 0$, then*

$$\rho_E \left(\int_t^1 g(u) \frac{du}{u} \right) \lesssim \rho_E(g), \quad g \in D_0.$$

Proof We are going to use Minkowski's inequality for the equivalent p -norm of ρ_E . To this end, first we replace the integral by sums using monotonicity properties of $g \in D_0$.

Thus

$$\begin{aligned} \int_t^1 g(u) \frac{du}{u} &\lesssim \int_1^\infty \chi_{(0,1)}(tv) g(tv) \frac{dv}{v} \\ &= \sum_{l=0}^\infty \int_{2^l}^{2^{l+1}} \chi_{(0,1)}(tv) g(tv) \frac{dv}{v} \\ &\lesssim \sum_{l=0}^\infty \chi_{(0,1)}(t2^l) g(t2^l). \end{aligned}$$

Applying Minkowski's inequality, we get

$$\begin{aligned} \rho_E^p \left(\int_t^1 g(u) \frac{du}{u} \right) &\lesssim \sum_{l=0}^\infty \rho_E^p (\chi_{(0,1)}(t2^l) g(t2^l)) \\ &\lesssim \rho_E^p(g) \sum_{l=0}^\infty h_E^p(2^{-l}) \\ &\lesssim \rho_E^p(g) \sum_{l=0}^\infty 2^{-l(\alpha_E - \varepsilon)}. \end{aligned}$$

Choosing $\varepsilon > 0$ such that $\alpha_E > \varepsilon$, we conclude the proof. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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